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THE MOTION OF A BODY WITH ELASTIC ELEMENTS AROUND A FIXED POINT IN ACTION-ANGLE VARIABLES†

L. I. KONKINA

Velikiye Luki

e-mail: vgsha@mart.ru

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The motion of a mechanical system formed by a rigid body rotating around a fixed point and carrying elastic rods, which undergo flexural deformation, is investigated. The asymptotic method of constructing approximate equations describing the evolution of the system motion in canonical action-angle variables is employed. The method of separating the motions and using the averaging operator enables the qualitative features of the behaviour of the system to be investigated, since, as a rule, the equations of motion of such systems cannot be integrated in explicit form. © 2000 Elsevier Science Ltd. All rights reserved.

1. Consider an asymmetrical body with elastic elements in the form of two pairs of rods, situated in the equatorial plane of the ellipsoid of inertia, rotating around a fixed point O. The rods are flexible, and the flexural deformations are accompanied by energy dissipation.

We will introduce two Cartesian systems of coordinates: the OX_k axes ($k = 1, 2, 3$) are fixed, the OX_3, axis is directed vertically upwards, and the OZ_k axes are connected with the principal axes of inertia of the body, where OZ_3 is the axis of natural rotation of the body. In the OZ_1Z_2 plane two pairs of flexible rods are placed along the principal axes of the ellipsoid of inertia OZ_1 and OZ_2. The rods are deformed during the motion. The potential energy functionals of elastic deformations and dissipative forces are found from the formulae

$$E[\mathbf{u}] = \frac{N}{2}\int_V \sum_{i=1}^{2}\sum_{j=1}^{3}\left(\frac{\partial^2 u_{ij}}{\partial s^2}\right)^2 ds, \quad D[\dot{\mathbf{u}}] = \chi b E[\dot{\mathbf{u}}] \tag{1.1}$$

where N is the bending stiffness of the rod, χ is a constant characterizing the energy dissipation in the rod on bending, b is a dimensional constant, and $u_{ij}(s, t)(i = 1, 2; j = 1, 2, 3)$ is the deviation of the cross-section of one of the rods with coordinate s on bending with respect to the corresponding axis. The vector $\mathbf{u}(s, t)$ is taken to be equal to $\mathbf{u}_1$ and $\mathbf{u}_2$.

We will write the equations of motion of the system in the form

$$\begin{aligned} &\dot{I} = -\nabla_w R(I, w, u, \dot{u}), \quad \dot{w} = \nabla_I R(I, w, u, \dot{u}) \\ &\frac{d}{dt}\nabla_{\dot{u}} R - \nabla_u R = -Q_u, \quad Q_u = -\frac{\partial}{\partial \dot{u}} D[\dot{\mathbf{u}}] \end{aligned} \tag{1.2}$$

where Q_u are the dissipative forces.

The Routh functional $R(I, w, u, \dot{u})$, describing the motion of the system, is given by the formula

$$\mathbf{R} = \frac{1}{2}[\mathbf{G} - \mathbf{G}_u, \; J^{-1}[\mathbf{u}](\mathbf{G} - \mathbf{G}_u)] - \frac{1}{2}\int_V (\mathbf{u}_1^2 + \dot{\mathbf{u}}_2^2)\rho ds + E[\mathbf{u}] \tag{1.3}$$

$$\mathbf{G}_u = \int_K \sum_{i=1}^{2} (\mathbf{r} \times \dot{\mathbf{r}})\rho ds$$

where $\mathbf{G}$ is the angular momentum vector.

Using the linear theory of the bending of think rectilinear rods, the radius vector of a point of the rod in the OZ_k system can be represented in the form [2, 3]

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$$
\begin{aligned}
&\mathbf{r}_1+\mathbf{u}_1 = s\mathbf{e}_1 + u_{12}(s,t)\mathbf{e}_2 + u_{13}(s,t)\mathbf{e}_3, \quad \mathbf{r}_1 = s\mathbf{e}_1 \\
&\mathbf{r}_2+\mathbf{u}_2 = u_{12}(s,t)\mathbf{e}_1 + s\mathbf{e}_2 + u_{23}(s,t)\mathbf{e}_3, \quad \mathbf{r}_2 = s\mathbf{e}_2 \\
&s \in V = [-l,-a]\cup[a,l]
\end{aligned} \tag{1.4}
$$

where $\mathbf{e}_k$ (k = 1, 2, 3) is the unit vector of the OZ_k axis, l are constants, and ρ is the density of the rod material, assumed to be uniform.

The inertia tensor $J[\mathbf{u}]$ of the system, consisting of the rigid body and the deformed rods, is considered in a moving system of coordinates. We will write the inertia tensor assuming that u_{ij} are small quantities, and hence we can confine ourselves to terms that are linear in u_{ij}. We obtain

$$
\begin{aligned}
&J^{-1}[\mathbf{u}] \approx J_0^{-1} - J_0^{-1}J_1J_0^{-1} \\
&J_0 = \mathrm{diag}\{A,B,C\}, \quad A = A_1 + f[\mathbf{u}_2], \quad B = B_1 + f[\mathbf{u}_1], \quad C = C_1 + f[\mathbf{u}_1] + f[\mathbf{u}_2] \\
&f[\mathbf{u}_i] = \int_{-l}^{l} (2su_i + u_i^2)\rho ds, \quad (i = 1,2)
\end{aligned}
$$

(J_1 is the component of the inertia tensor linear with respect to $\mathbf{u}$).

We will represent the Routh function as follows

$$
R = R_0 + \varepsilon R_1 + \ldots, \quad \varepsilon = bN^{-1}
$$

where R_0 is a function describing the unperturbed motion of the body, assuming that the rods are not deformed ($\mathbf{u}$=0) and ε is a small dimensionless parameter.

2. The unperturbed motion (ε = 0) of an asymmetrical body ($A \neq B \neq C$) is described by the equations of rotation of a solid in the Euler case. We will use the canonical action-angle variables I_1, I_2, I_3, w_1, w_2, w_3 as the variables [1]. When introducing the action-angle variables the initial variables are the Andoyer variables L, I_2, I_3, 1, φ_2, φ_3. When ε = 0 the modulus of the kinetic moment I_2, and also the canonically conjugate variables I_3, φ_3, which specify the position of this vector in the OX_k axes, are constant quantities. We will introduce the following parameters into the unperturbed problem

$$
\chi^2 = \frac{C(A-B)}{A(B-C)}, \quad \lambda^2 = \chi^2 \frac{A}{C}\frac{2Ch - I_2^2}{I_2^2 - 2Ah} \tag{2.1}
$$

where h is the energy constant. The values λ = 0, $\lambda = \infty$, λ = 1 correspond to rotations of the body around the major axis (OX_3), the minor axis (OX_1) and the middle axis (OX_2) of the ellipsoid of inertia.

We will investigate the problem in the regions of rotational motions ($\lambda \leqslant 1$) with the condition that the rotation frequencies of the body are incommensurable, i.e. $\mathbf{k}\omega_k = 0$ (k = 1, 2), where $\mathbf{k}$ is an integral vector, is only satisfied when $\mathbf{k}$ = 0. In these regions of motion the action-angle variables are given by the formulae

$$
\begin{aligned}
&I_1 = \frac{2I_2}{\pi\chi}\sqrt{\frac{1+\chi^2}{\chi^2+\lambda^2}}[(\chi^2+\lambda^2)\Pi(\chi^2,\lambda) - \lambda^2 K(\lambda)] \\
&I_2 = \pm\frac{\sqrt{\chi^2+\lambda^2}}{\chi}\left[\mathrm{dn}\left(\frac{2K(\lambda)}{\pi}w_1,\lambda\right)\right]^{-1}, \quad I_3 = I_3
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
&w_1 = \frac{\pi}{2K(\lambda)}F(\xi,\lambda), \quad w_2 = \varphi_2 + \frac{i}{2}\ln\frac{\theta_4(w_1 - i\sigma)}{\theta_4(w_1 + i\sigma)} \\
&\xi = \pm\mathrm{am}(\tau_1,\lambda), \quad \sigma = \frac{\pi}{2K(\lambda)}F\left(\mathrm{arctg}\frac{\chi}{\lambda}, \lambda'\right) \\
&\tau_1 = \frac{2K(\lambda)}{\pi}w_1, \quad \chi^2 = \frac{C(A-B)}{A(B-C)}, \quad \lambda^2 = 2\chi^2\frac{I_2 - I_1}{I_2\sqrt{1+\chi^2}}
\end{aligned}
$$

where $K(\lambda)$, $\Pi(\varkappa, \lambda)$ are the complete elliptic integrals of the first and third kind respectively, $F(\xi, \lambda)$ is the incomplete elliptic integral of the first kind, θ_4 is the Jacobi theta function, and $\lambda(I_1, I_2)$ is the single-valued solution of the equation for I_1 from (2.2) with respect to λ.

We will also write the Routh function for unperturbed motion for arbitrary A, B and C in action-angle variables in the form

$$R_0 = \frac{I_2^2}{2A}\left[1 - \frac{C-A}{A}\frac{\chi^2}{\chi^2+\lambda^2}\right]$$

The canonical equations of unperturbed motion with the function R_0 can be integrated, and the general solution has the form

$$I_i = I_{i_0}; \quad w_1 = \omega_1 t + w_1^0, \quad w_2 = \omega_2 t + w_2^0, \quad w_3 = w_3^0 \tag{2.3}$$

The frequencies of unperturbed motion are given by the following expressions [4]

$$\omega_1 = -\frac{C-A}{AC}\frac{\sqrt{1+\chi^2}I_2^3}{\varkappa^2}, \quad \omega_2 = \frac{I_2^2}{A} - \frac{C-A}{2AC}\frac{I_1 I_2^2}{\varkappa}; \quad \varkappa = \sqrt{1+\chi^2}I_2^2 + 2I_1 \tag{2.4}$$

Flexural vibrations of the rods in the perturbed motion are described by the second of the equations of system (1.1)

$$\begin{aligned}&\rho\frac{\partial^2 u_{ij}}{\partial t^2} + N\frac{\partial^4 u_{ij}}{\partial s^4} + \chi N\frac{\partial^5 u_{ij}}{\partial t\partial s^4} + \frac{d}{dt}[J^{-1}[\mathbf{u}](\mathbf{G}-\mathbf{G}_u)(\mathbf{r}_i+\mathbf{u}_i)]\rho\mathbf{e}_j + \\ &+\frac{1}{2}((\mathbf{G}-\mathbf{G}_u)\nabla_{u_{ij}}J^{-1}[\mathbf{u}](\mathbf{G}-\mathbf{G}_u)) + [J^{-1}[\mathbf{u}](\mathbf{G}-\mathbf{G}_u)\dot{\mathbf{u}}\rho]\mathbf{e}_j = 0, \quad i=1,2; \\ &j=1,2,3; \quad i\neq j\end{aligned} \tag{2.5}$$

The boundary conditions for the funtion $u_{ij}(s, t)$ are given in the form

$$u_{ij}(\pm a, t) = \frac{\partial u_{ij}(\pm a, t)}{\partial s} = 0, \quad \frac{\partial^2 u_{ij}(\pm l, t)}{\partial s^2} = \frac{\partial^3 u_{ij}(\pm l,\ t)}{\partial s^3} = 0$$

The solution describing the forced vibrations of the rods will be sought in the form of a series in the small parameter $\varepsilon = N^{-1}$

$$u_{ij} = \varepsilon u_{ij}^{(1)} + \varepsilon^2 u_{ij}^{(2)} + \ldots, \quad i=1,2; \quad j=1,2,3; \quad i\neq j$$

The functions u_{ij} satisfy the equation

$$\left(1+\chi b\frac{\partial}{\partial t}\right)\frac{\partial^4 u_{ij}^{(1)}}{\partial s^4} + \frac{1}{2}(\mathbf{G}, \nabla_{\mathbf{u}_{ij}}J^{-1}[\mathbf{u}]\mathbf{G}) + \frac{d}{dt}[J^{-1}[\mathbf{u}]\mathbf{G}\times(\mathbf{r}_i+\mathbf{u}_i)]\rho\mathbf{e}_j = 0 \tag{2.6}$$

The action I – angle w equations (2.6) can be written in canonical variables as follows:

$$\frac{\partial^4 u_{ij}^{(1)}}{\partial s^4} + \chi b\frac{\partial^5 u_{ij}^{(1)}}{\partial t\,\partial s^4} = f_{ij}s, \quad i=1,2; \quad j=1,2,3; \quad i\neq j \tag{2.7}$$

The functions f_{ij} are defined by the relations

$$f_{12} = -\Sigma b_{n,0}b'_{n,0}\sin 2nw_1, \quad f_{13} = \Sigma b_{n,0}F\cos nw_1, \quad f_{23} = \Sigma b'_{n,0}F\sin nw_1$$

$$F = \left[\frac{\pi I_2}{2K(\lambda)}\frac{A-C}{AC}\frac{\chi}{\sqrt{(1+\chi^2)(\lambda^2+\chi^2)}}\right]$$

$$b_{n,0} = \frac{I_2}{A}\frac{\pi}{K(\lambda)\sqrt{\lambda^2+\chi^2}}\frac{q^{m+1/2}}{1+q^{2m+1}}, \quad b'_{n,0} = \frac{I_2}{B}\frac{i\pi}{K(\lambda)}\sqrt{\frac{\chi^2+1}{\lambda^2+\chi^2}}\frac{q^{m+1/2}}{1-q^{2m+1}}$$

$$q = \exp[-\pi K'(\lambda)/K(\lambda)]$$

where $n = 2m+1$, and the summation is carried out over all integer m from $-\infty$ to $+\infty$.

Suppose u_{ij0} is the solution of Eq.(2.7) for $\chi = 0$. Then, the particular solution describing forced flexural vibrations of the rods has the form

$$u_{ij}^{(1)} = \sum_{k=0}^{\infty} (-\chi)^k \frac{\partial^k u_{ij}^{(1)}}{\partial t^k}$$

$$u_{ij}^{(1)} = f_{ij}\psi(s), \quad \psi(s) = \frac{s^5}{120} - \frac{l^2 s^3}{12} + \frac{l^3 s^2}{6} \operatorname{sign} s \tag{2.8}$$

The function $\psi(s)$ is determined for the boundary conditions

$$u_{ij0}^{(1)}(0,t) = \frac{\partial}{\partial s} u_{ij0}^{(1)}(0,t) = 0$$

(the clamping conditions at the origin of coordinates, where for simplicity $a = 0$) and the dynamic boundary conditions

$$\frac{\partial^2}{\partial s^2} u_{ij0}^{(1)}(\pm l,t) = \frac{\partial^3}{\partial s^3} u_{ij0}^{(1)}(\pm l,t) = 0$$

Note that the solutions of Eq.(2.7) are found by the method of separation of the motions, provided that the canonical action-angle variables correspond to the unperturbed problem.

Confining ourselves in series (2.8) to the first two terms, we have

$$u_{ij}^{(1)} = u_{ij0}^{(1)} - \chi \dot{u}_{ij0}^{(1)} \tag{2.9}$$

We will write the equations of perturbed motion of the system, substituting the displacement $u_{ij} = \varepsilon\, u_{ij}^{(1)}$ into Eqs (1.2)

$$\begin{aligned}
\dot{I}_1 &= -\nabla_{w_1} = -\Sigma b_{n,0} n \sin n w_1 \int s\dot{u}_{23}\rho ds - \Sigma b'_{n,0} n \cos w_1 \int s\dot{u}_{13}\rho ds - \\
&-\Sigma b_{n,0} b'_{n,0} \sin 2n w_1 \int s(u_{12} + u_{21})\rho ds + \Sigma n b_{n,0} \dot{w}_1 \sin n w_1 \int s u_{13}\rho ds + \\
&+\Sigma n b'_{n,0} \dot{w}_1 \cos n w_1 \int s u_{23} \rho ds
\end{aligned} \tag{2.10}$$

$$\dot{I}_i = -\nabla_{w_i} R = 0, \quad i = 2,3; \quad \dot{w}_j = \nabla_{I_j} R, \quad j = 1,2,3$$

It follows from these equations that the value of the kinetic moment vector I_2 and its projection onto the OX_3 axis retain their values, whereas the action variable I_1 depends on the elastic vibrations and the dissipative forces, leading to the evolution of the system.

3. Applying the averaging operator with respect to the angular variables w_1 and α to Eqs(2.10), we obtain, after calculation, the following averaged system of equations

$$\langle \dot{I}_1 \rangle = \Phi(s) \left\{ \chi \left[\left(\frac{A-C}{AC} \frac{\pi l_2 \chi n}{K(\lambda)\sqrt{(1+\lambda^2)(\lambda^2+\chi^2)}} \right)^2 \Sigma(b_{n,0}^2 - b_{n,0}'^2) \right] - \Sigma(b_{n,0}^2 - b_{n,0}'^2) \right\} \tag{3.1}$$

$$\langle \dot{I}_i \rangle = 0, \quad i = 2,3; \quad \Phi(s) = \int \psi(s) \rho ds$$

We will obtain the steady solutions of Eq.(3.1) assuming that $\langle I_1 \rangle = 0$. Up to terms in λ^2 we obtain

$$\Phi(s)\chi \left(\frac{A-C}{AC} \frac{\sqrt{1+\chi^2 I_2^3}}{[\sqrt{1+\chi^2 I_2} + 2I_1]^2} \right)^2 \Sigma(b_{n,0}^2 - b_{n,0}'^2) - \Sigma b_{n,0}^2 b_{n,0}'^2 = 0 \tag{3.2}$$

We will write the coefficients of the first terms of the expansion occurring in expression (3.2)

$$b_{00}=\frac{2}{4+\lambda^2}\left(\frac{2\chi^2+\lambda}{\chi^2}\right),\quad b_{1,0}=\frac{\lambda}{4+\chi\lambda^2},\quad b_{2,0}=\frac{16(1-\lambda)}{\chi(\lambda-\chi^2)}$$

$$b'_{0,0}=\frac{(2\chi^2+\lambda^2)(2+\chi^2)}{\chi(4+\lambda^2)},\quad b'_{1,0}=\frac{(2+\chi)\lambda}{2\chi^2(4+\lambda^2)},\quad b'_{2,0}=\frac{8(2+\chi^2)(1-\lambda)}{\chi(\lambda-\chi^2)}$$

Substituting into these coefficients the expression

$$\lambda^2=2\chi^2\frac{I_2-I_1}{I_2\sqrt{1+\chi^2}}$$

which is the single-valued solution of the equation for I_1 (up to terms in λ^2), which occurs in (2.2), after calculation we obtain expressions for the coefficients, which depend explicitly on the action variables I_i.

We will write relation (3.2) as follows:

$$\Phi(s)\{\chi\}\,\dot{w}_1^2Q-Q_1=0$$

Hence, we can obtain the value of the angular velocity of rotation of the body around the natural axis of rotation

$$\omega_1=\sqrt{\frac{Q_1}{\chi\Phi(s)Q}};\quad Q=\Sigma(b_{n,0}^2-b'^2_{n,0}),\quad Q_1=\Sigma b_{n,0}^2 b'^2_{n,0}$$

It can be seen that the limit values for the action variables and the angular velocity of rotation around the axis of natural rotation depend on the deformations of the rods and the dissipative forces.

By considering the case of a dynamically symmetrical body ($A = B$) in this formulation of the problem, we obtain that, at the stage of the first approximation, the evolution of the motion is identical with the results obtained in similar investigations in Andoyer canonical variables [2, 3].

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